MATH 307

Hw 3
Ch. 3 Intro sheet
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Ch. 3 2nd order intro

A 2nd order ODE looks like

$$y'' = f(t, y, y')$$

We say it is linear if it can be written as

$$y'' + p(t)y' + q(t)y = g(t)$$

We say it is linear with constant coefficients if it can be written as

$$ay'' + by' + cy = g(t)$$

We say it is homogeneous if $g(t) = 0$

NOTES:

- We will not discuss Non linear.
- We will primarily discuss constant coeff.
- No slope field to draw!
- There will be two unknown constant in our general sol'n.

Ex)

$$y'' = -9.8$$

$$y' = -9.8t + c_1$$

$$y = -4.9t^2 + c_1 t + c_2$$

2 constants

Two initial conditions $\begin{cases} y(t_0) = y_0 \\ y'(t_0) = y_0' \end{cases}$

3.1 : Linear Homogeneous Constant Coefficient with Two Real Roots

Consider

$$ay'' + by' + cy = 0$$

Observe: Sol'ns will need to be functions where derivatives "cancel" the function value

- CAN'T BE $y = t^3$, $y' = 3t^2$, $y'' = 6t$
- CAN'T BE $y = \ln(t)$, $y' = \frac{1}{t}$, $y'' = \frac{-1}{t^2}$
- COULD BE $y = e^{-2t}$, $y' = -2e^{-2t}$, $y'' = 4e^{-2t}$
  or $y = \sin(x)$, $y' = \cos(x)$, $y'' = -\sin(x)$

We will see that sol'ns involve e^{rt}, $\cos(rx)$, and/or $\sin(rx)$

TODAY: Consider $y = e^{rt}$

$$y' = re^{rt}$$
$$y'' = r^2 e^{rt}$$

Ex) $y = e^{5t}$
$y' = 5e^{5t}$
$y'' = 25e^{5t}$

IF $y = e^{rt}$ is a sol'n to $ay'' + by' + cy = 0$

THEN $ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$ for all t

$$\underbrace{e^{rt}}_{\text{NEVER ZERO}}(ar^2 + br + c) = 0 \quad \text{for all } t$$

$$\boxed{ar^2 + br + c = 0}$$

Characteristic Equation

$$r_1 = \frac{-b - \sqrt{b^2-4ac}}{2a} \quad , \quad r_2 = \frac{-b + \sqrt{b^2-4ac}}{2a}$$

- $\Delta = b^2 - 4ac > 0 \Rightarrow$ TWO REAL ROOTS $\Rightarrow$ TODAY!
- $\Delta = b^2 - 4ac < 0 \Rightarrow$ NO REAL ROOTS
  TWO IMAGINARY ROOTS
- $\Delta = b^2 - 4ac = 0 \Rightarrow$ ONE REAL ROOT
  (REPEATED ROOT)

The general sol'n to $ay'' + by' + cy = 0$

IS $\boxed{y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}}$

Ex Find the general sol'n to

$$y'' - 7y' + 10y = 0$$

$$r^2 - 7r + 10 \stackrel{?}{=} 0$$

$$(r-2)(r-5) = 0$$

$$r_1 = 2, \quad r_2 = 5$$

$y_1(t) = e^{2t}$ ONE SOL'N, $y_2(t) = e^{5t}$ ANOTHER SOL'N

check
$25e^{5t} - 35e^{5t} + 10e^{5t} = 0$ ✓

General Sol'n: $\boxed{y(t) = c_1 e^{2t} + c_2 e^{5t}}$

INITIAL CONDITIONS

- Substitute them in and solve for c_1 and c_2.

Ex) $y'' - 4y = 0 \qquad y(0) = 1, \; y'(0) = 8$

gen sol'n

$r^2 - 4 = 0$

$(r+2)(r-2) = 0 \qquad r_1 = -2, \; r_2 = 2$

$y_1(t) = e^{-2t} \qquad y_2(t) = e^{2t}$

$y(t) = c_1 e^{-2t} + c_2 e^{2t}$

$y'(t) = -2c_1 e^{-2t} + 2c_2 e^{2t}$

initial conditions

$y(0) = 1 \Rightarrow c_1 + c_2 = 1 \qquad (i)$

$y'(0) = 8 \Rightarrow -2c_1 + 2c_2 = 8 \qquad (ii)$

(i) $c_1 + c_2 = 1$

(ii) $-c_1 + c_2 = 4$

$2c_2 = 5$

$\boxed{c_2 = 5/2}$

$\boxed{c_1 = 1 - c_2 = -3/2}$

OR

(i) $c_2 = 1 - c_1$

(i) & (ii) $-2c_1 + 2(1 - c_1) = 8$

$-4c_1 + 2 = 8$

$-4c_1 = 6$

$c_1 = -6/4 = -3/2$

$c_2 = 1 - c_1 = 5/2$

$$y(t) = -\frac{3}{2}e^{-2t} + \frac{5}{2}e^{2t}$$

Note: $\lim_{t\to\infty} e^{rt} = \begin{cases} 0, & \text{if } r<0; \\ 1, & \text{if } r=0; \\ \infty, & \text{if } r>0. \end{cases}$

So
- if $r_1<0$ AND $r_2<0$, then $\lim_{t\to\infty} y(t)=0$
- if $r_1>0$ or $r_2>0$ (and coeff. is not zero) $\lim_{t\to\infty} y(t) = \pm\infty$ ← depending on coeff.
- if $r_1=0$ or $r_2=0$

  the limit could be something else

Ex) $y''+7y'=0$

$r^2+7r=0$
$r(r+7)=0$
$r_1=0,\ r_2=-7$

$y(t)=c_1e^{0t}+c_2e^{-7t}$

$\boxed{y(t)=c_1+c_2e^{-7t}}$ $\lim_{t\to\infty} y(t)=c_1$

$y''-5y'=0$

$r^2-5r=0$
$r(r-5)=0$
$r_1=0,\ r_2=5$

$\boxed{y(t)=c_1+c_2e^{5t}}$ $\lim_{t\to\infty} y(t)=\begin{cases}\infty, & \text{if } c_2\neq 0; \\ c_1, & \text{if } c_2=0.\end{cases}$

STOP

Ex) $y'' + 5y' + 6y = 0 \qquad y(0) = 2, \; y'(0) = 3$

$$r^2 + 5r + 6 = 0$$
$$(r+2)(r+3) = 0 \qquad r_1 = -2, \; r_2 = -3$$

$$\boxed{y(t) = c_1 e^{-2t} + c_2 e^{-3t}} \qquad y'(t) = -2c_1 e^{-2t} - 3c_2 e^{-3t}$$

$y(0) = 2 \Rightarrow \quad c_1 + c_2 = 2 \qquad$ (i) $\quad 2c_1 + 2c_2 = 4$

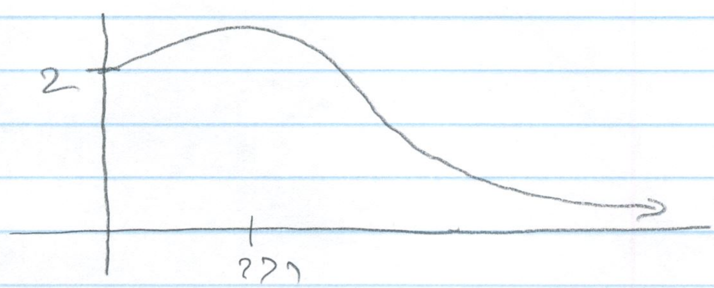
$y'(0) = 3 \Rightarrow -2c_1 - 3c_2 = 3 \qquad$ (ii) $\quad -2c_1 - 3c_2 = 3$

$$-c_2 = 7$$
$$c_2 = -7$$
$$c_1 = 9$$

$$\boxed{y(t) = 9e^{-2t} - 7e^{-3t}}$$

$\rightarrow$ GRAPH

$\lim_{t \to \infty} y(t) = 0$



WHERE IS MAX?

$y'(t) = -18e^{-2t} + 21e^{-3t}$

$$-18e^{-2t} + 21e^{-3t} \stackrel{?}{=} 0$$
$$21e^{-3t} = 18e^{-2t}$$
$$\frac{21}{e^{3t}} = \frac{18}{e^{2t}}$$
$$\frac{21}{18} = \frac{e^{3t}}{e^{2t}}$$
$$\frac{7}{6} = e^{t}$$

$t = \ln(7/6) \approx 0.15415$

$y = y(\ln(7/6)) \approx 2.20408$